Function Theory of a Complex Variable (E2): Exercise sheet 3 solutions

1. We have that
$$2^i = e^{i(\log 2 + i\arg 2)} = e^{-2n\pi + i\log 2}, \qquad n \in \mathbb{Z}.$$
Similarly,
$$i^i = e^{i(\log 1 + i\arg i)} = e^{-\pi(\frac{1}{2}+2n)}, \qquad n \in \mathbb{Z}.$$
Moreover,
$$(-1)^{2i} = e^{2i(\log 1 + i\arg(-1))} = e^{-2\pi(1+2n)}, \qquad n \in \mathbb{Z}.$$

2. We have that
$$\sqrt{1+z} + \sqrt{1-z} = e^{\frac{1}{2}\log(1+z)} + e^{\frac{1}{2}\log(1-z)}.$$
Using the principal branch of log, this is single-valued in
$$\mathbb{C}\backslash\{z \in \mathbb{R} : \ z \in (-\infty, -1] \cup [1, \infty)\}.$$
In this region, it has derivative:
$$\frac{1}{2}\left(\frac{e^{\frac{1}{2}\log(1+z)}}{1+z} - \frac{e^{\frac{1}{2}\log(1-z)}}{1-z}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{1+z}} - \frac{1}{\sqrt{1-z}}\right),$$
and so is analytic.

3. Taking $z(t) = t(1+i)$, we find that
$$\int_\gamma x dz = \int_0^1 \mathrm{Re}(z(t))z'(t)dt = \int_0^1 t(1+i)dt = \frac{1+i}{2}.$$

4. We have that
$$\int_{\partial B(0,2)} \frac{1}{z^2-1}dz = \frac{1}{2}\left(\int_{\partial B(0,2)} \frac{1}{z-1}dz - \int_{\partial B(0,2)} \frac{1}{z+1}dz\right).$$
Now, similarly to the example in lectures, we have
$$\int_{\partial B(0,2)} \frac{1}{z-1}dz = \lim_{\varepsilon\to 0}\int_{-(\pi-\varepsilon)}^{\pi-\varepsilon} \frac{2ie^{i\theta}}{2e^{i\theta}-1}d\theta = \lim_{\varepsilon\to 0}\left[\log(2e^{i\theta}-1)\right]_{-(\pi-\varepsilon)}^{\pi-\varepsilon} = 2\pi i.$$
And a similar calculation yields:
$$\int_{\partial B(0,2)} \frac{1}{z+1}dz = \lim_{\varepsilon\to 0}\int_{-(\pi-\varepsilon)}^{\pi-\varepsilon} \frac{2ie^{i\theta}}{2e^{i\theta}+1}d\theta = \lim_{\varepsilon\to 0}\left[\log(2e^{i\theta}+1)\right]_{-(\pi-\varepsilon)}^{\pi-\varepsilon} = 2\pi i.$$
Hence,
$$\int_{\partial B(0,2)} \frac{1}{z^2-1}dz = 0.$$

5. Define

$$f(z) := \frac{e^z - 1}{z}.$$

Then f is analytic in $\mathbb{C}\backslash\{0\}$, and $zf(z) \to 0$ as $z \to 0$. Thus

$$\int_{\partial B(0,1)} f(z)dz = 0.$$

This implies

$$\int_{\partial B(0,1)} \frac{e^z}{z} dz = \int_{\partial B(0,1)} \frac{1}{z} dz = 2\pi i.$$

Since $z^{-1}e^z$ is analytic in $B(2, \frac{3}{2})$, we immediately obtain from Cauchy's theorem that

$$\int_{\partial B(2,1)} \frac{e^z}{z} dz = 0.$$

6. We have that

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where the power series has an infinite radius of convergence. Define

$$g(z) := \frac{f(z) - a_0 - a_1 z - \cdots - a_{n-1} z^{n-1}}{z^n} = \sum_{k=0}^{\infty} a_{n+k} z^k.$$

Now, the radius of convergence of sum above is given by

$$\left(\limsup_{k\to\infty} \sqrt[k]{|a_{n+k}|}\right)^{-1} = \left(\limsup_{k\to\infty} (\sqrt[k]{|a_k|})^{\frac{k}{k-n}}\right)^{-1} = \infty,$$

since $\sqrt[k]{|a_k|} \to 0$ and $\frac{k}{k-n} \to 1$. Hence g is analytic everywhere. Moreover, by the assumption on f, there exists a constant $C < \infty$ such that

$$|g(z)| \le \frac{|f(z)| + |a_0| + |a_1 z| + \cdots + |a_{n-1} z^{n-1}|}{|z|^n} \le C, \qquad \forall |z| \ge R.$$

Since g is analytic (and therefore continuous), it follows that g is bounded on $\mathbb{C}$. Hence, Liouville's theorem gives us that g is constant, and the result that f is a polynomial (of degree at most n) follows.

7. For $r \in (0, 1)$, Cauchy's inequality yields

$$|f^{(n)}(0)| \le \frac{n! \sup_{z:\,|z|=r} |f(z)|}{r^n} = \frac{n!}{(1-r)r^n}.$$

The optimal r is easily checked to be $r = \frac{n}{n+1}$, using which we obtain

$$|f^{(n)}(0)| \le (n+1)!(1+n^{-1})^n.$$